Modulational Instability and Stationary Waves for the Coupled Generalized Schrödinger-Boussinesq System

Zu-Feng Liang

Department of Physics, Hangzhou Normal University, Hangzhou, 310036, P. R. China

Reprint requests to Z. L.; E-mail: liangzufeng@163.com

Z. Naturforsch. **66a**, 143 – 150 (2011); received April 22, 2010

The coupled generalized Schrödinger-Boussinesq (SB) system, which can describe a high-frequency mode coupled to a low-frequency wave in dispersive media is investigated. First, we study the modulational instability (MI) of the SB system. As a result, the general dispersion relation between the frequency and the wave number of the modulating perturbations is derived, and thus a number of possible MI regions are identified. Then two classes of exact travelling wave solutions are obtained expressed in the general forms. Several explicit examples are presented.

Key words: The Nonlinear Schrödinger-Boussinesq Equation; Modulational Instability; Solitary Waves.

1. Introduction

The coupled Schrödinger-Boussinesq (SB) system can govern the coupled wave propagation in nonlinear dispersive media wherein an amplitude modulated high-frequency wave is coupled to a suitable low-frequency eignemode of the medium. For instance, the SB equations were derived to govern the stationary propagation of coupled upper-hybrid and magnetoacoustic waves in a magnetized plasma [1], where the generic Hamiltonian of the SB was presented. It was shown that the nonlinear propagation of coupled Langmuir and ion-acoustic waves in a two-electron temperature plasma could also be governed by the generalized SB system [2], where a new class of coupled Langmuir-ion-acoustic solitons propagating with supersonic speeds but accompanied by density rarefactions was found. Later, nonlinear propagation of intense electromagnetic waves in a hot electron-positron relativistic plasma containing a small fraction of cold electron-ion component was investigated by the generalized SB system [3].

In the present paper, we consider the following coupled generalized Schrödinger-Boussinesq (SB) system

$$\mathrm{i}(E_t+\lambda_1 E_x)+\lambda_2 E_{xx}+\lambda_4 E=\lambda_3 NE, \tag{1}$$

$$N_{tt}+\mu_1 N_{xx}+\mu_2 N_{xxxx}+\mu_3 (N^2)_{xx}=\mu_4(|E|^2)_{xx}, \tag{2}$$

where μ_i and λ_i ($i=1,2,3,4$) are real constants. It is noted that a more general system could be introduced by adding a cubic term in (2) arising from the self-interaction of the waves [3,4] or in (1) by considering a higher-order nonlinearity [2,3]. On the other hand, this system can be reduced to the nonlinear Schrödinger (NLS) equation and the Zakharov equation. Many solutions for the generalized SB system for an appropriate choice of parameters have been reported [1–3,5–8]. For instance, the homoclinic solution of the SB system with the parameters $\lambda_1=0$, $\lambda_2=\lambda_3=\mu_3=1$, $\mu_2=-1/3$, $\mu_4=1/3$ has been obtained via the bilinear method [5]. By means of the same method, one-soliton solution exists for the SB equations if $\lambda_1=\lambda_4=0$, $\lambda_2=-1$, $\lambda_3=1$, $\mu_3=3\mu_2$, and $\mu_4=\mu_1$. If one assumes $\mu_3=-1$ further, then the system allows an N-soliton solution [6]. The Painlevé analysis of the SB system has been carried out for $\lambda_4=0$ [8], where two branches of leading singularities were identified, and it was revealed that the system is completely integrable in one of the branches, for which the associated Bäcklund transformation was explicitly given. The existence and the orbital stability of solitary waves of the SB system for $\lambda_1=\lambda_4=0$, $\lambda_2=\mu_2=1$, $\lambda_3=\mu_1=\mu_4=-1$, $\mu_3=3$ has also been investigated [9].

In this paper, we are focused on the generalized SB system of (1) and (2) to investigate their modulational instabilities and possible new exact solutions. The paper is organized as follows. In Section 2, we investigate the linear modulational instability of the SB system. A general nonlinear dispersion relation associated with the frequency and wave number of the modulat-

0932–0784 / 11 / 0300–0143 $ 06.00

ing perturbations is derived. It is found that instability might arise in many different regions. In Section 3, by deforming to the ϕ^4 model, we obtain two general solutions constructed on the solutions of the ϕ^4 model. In detail, we present three types of exact periodic solutions expressed by Jacobian elliptic functions. Representative profiles of the waves are graphically displayed. In Section 4, we give a brief conclusion.

2. Modulational Instability Analysis

It is known that the modulational instability (MI) occurs in various fields such as plasmas, fluids, and nonlinear optics. It is the outcome of the interplay between the dispersion or diffraction effects with the nonlinearities. The MI of nonlinear waves has been studied extensively. For instance, some recent investigations are the MI of broadband optical pulses in a four-state atomic system governed by the generalized NLS equation [10], the interaction between nucleon and neutral scalar mesons governed by the coupled Schrödinger-Klein-Gordon equation [11, 12], the interaction of nonlinear dispersive waves on three channels, namely, laser beams on some dispersive material, modeled by the three coupled vector nonlinear Schrödinger equations [13].

Generally, one can analyze the modulational instability of a system through the following steps: (i) find an equilibrium state of the system; (ii) perturb the equilibrium state with a smaller perturbation wave number and frequency; (iii) derive a set of equations for the small perturbation functions, which will lead to the nonlinear dispersion relation; (iv) from the dispersion relation one can obtain a complex frequency revealing the growth of the amplitude modulated wave packet. From the above analysis, one can finally conclude if a wave under small perturbations moving along the system is stable or not.

An equilibrium state, namely, the simple and exact monochromatic wave solution of the SB equations, can be easily found by making the assumption

$$E = E_0 e^{i\omega t}, \quad N = N_0, \tag{3}$$

where constants ω, N_0 are real, and E_0 is complex. Substituting (3) into (1) – (2), we get

$$\omega = \lambda_4 - \lambda_3 N_0. \tag{4}$$

Next, we cause a small perturbation of the wave solution (3) in the form of

$$E = (E_0 + \varepsilon E_1) e^{i(\lambda_4 - \lambda_3 N_0)t}, \quad N = N_0 + \varepsilon N_1, \tag{5}$$

where E_1 is a complex quantity and N_1 is real. Then substituting (5) into (1) – (2), linearizing the result with respect to E_1 and N_1, writing $E_1 = u + iv$, $E_0 = a + ib$, $(u, v, a, b$ are real), and then separating the real and imaginary parts of the linearized equations (the first-order terms of ε), we finally obtain

$$u_t + \lambda_1 u_x + \lambda_2 v_{xx} - \lambda_3 b N_1 = 0, \tag{6}$$

$$v_t + \lambda_1 v_x - \lambda_2 u_{xx} + \lambda_3 a N_1 = 0, \tag{7}$$

and

$$\begin{aligned} &N_{1tt} + \mu_1 N_{1xx} + \mu_2 N_{1xxxx} + 2\mu_3 N_0 N_{1xx} \\ &-2a\mu_4 u_{xx} - 2b\mu_4 v_{xx} = 0. \end{aligned} \tag{8}$$

Now inserting $u = u_0 e^{i(Kx-\Omega t)} + \text{c.c.}$, $v = v_0 e^{i(Kx-\Omega t)} + \text{c.c.}$, and $N_1 = N_{10} e^{i(Kx-\Omega t)} + \text{c.c.}$, where K and Ω are the perturbation wave number and the frequency, respectively, and c.c. stands for the complex conjugate, into (6) – (8) yields a dispersion law for the perturbation wave

$$\begin{aligned} &\Omega^4 - 2\lambda_1 K \Omega^3 \\ &\quad - K^2[(\mu_2 + \lambda_2^2)K^2 - 2\mu_3 N_0 - \mu_1 - \lambda_1^2]\Omega^2 \\ &\quad + 2\lambda_1(\mu_2 K^2 - \mu_1 - 2\mu_3 N_0)K^3\Omega + \mu_2\lambda_2^2 K^8 \\ &\quad - (\mu_1\lambda_2^2 + 2\mu_3\lambda_2^2 N_0 + \mu_2\lambda_1^2)K^6 \\ &\quad - (2\mu_4\lambda_2\lambda_3|E_0|^2 - \mu_1\lambda_1^2 - 2\mu_3\lambda_1^2 N_0)K^4 = 0. \end{aligned} \tag{9}$$

It is noted that the coupled SB equations are modulationally stable for any wave number K if and only if four roots Ω of (9) are all positive real numbers. However, it is not so easy to find the roots of (9), since we have to employ the existing complicated analytical formulae and the associated criteria for the roots of a fourth-order polynomial. Therefore, we consider a special case, namely, $\lambda_1 = 0$, which simplifies (9) to

$$\Omega^4 - P\Omega^2 + Q = 0, \tag{10}$$

with P and Q given by

$$P = K^2\left[(\mu_2 + \lambda_2^2)K^2 - 2\mu_3 N_0 - \mu_1\right] \tag{11}$$

and

$$\begin{aligned} Q = \lambda_2 &\left[\mu_2\lambda_2 K^4 - (\mu_1 + 2\mu_3 N_0)\lambda_2 K^2 \right.\\ &\left. -2\mu_4\lambda_3|E_0|^2\right] K^4, \end{aligned} \tag{12}$$

respectively, and thus we get the solution

$$\Omega_\pm^2 = \frac{1}{2}\left[P \pm \sqrt{P^2 - 4Q}\right]. \tag{13}$$

In order to have positive real $\Omega_{\pm}^2$, it is easy to check that the following three conditions should be simultaneously satisfied:

$$P > 0, \quad Q > 0, \quad \Delta > 0, \tag{14}$$

where $\Delta = P^2 - 4Q$ is the discriminant quantity given by

$$\Delta = [(\mu_2 - \lambda_2^2)K^2 - \mu_1 - 2\mu_3 N_0]^2 + 8\lambda_2\lambda_3\mu_4|E_0|^2. \tag{15}$$

The first stability condition, $P > 0$, is satisfied for any $K \neq 0$ when $\mu_2 + \lambda_2^2 > 0$ and $2\mu_3 N_0 + \mu_1 < 0$. However, if $\mu_2 + \lambda_2^2 < 0$ and $2\mu_3 N_0 + \mu_1 > 0$, then P is always negative for any $K \neq 0$. Otherwise, $P = 0$ has two non-zero roots

$$K_{P\pm} = \pm\sqrt{\frac{2\mu_3 N_0 + \mu_1}{\mu_2 + \lambda_2^2}}, \tag{16}$$

and thus P is negative when $K \in (K_{P-}, K_{P+})$ for $\mu_2 + \lambda_2^2 > 0$, or $K \in (-\infty, K_{P-}) \cup (K_{P+}, +\infty)$ for $\mu_2 + \lambda_2^2 < 0$. In the case of $P < 0$, we either have $\Omega_-^2 < 0 < \Omega_+^2$ or $\Omega_-^2 < \Omega_+^2 < 0$ depending on the sign of Q.

Let us consider the second stability condition, $Q > 0$, in detail. We see that $Q = 0$ has two non-zero roots for K^2, namely,

$$K_{Q\pm}^2 = \frac{1}{2\mu_2\lambda_2}\left\{(\mu_1 + 2\mu_3 N_0)\lambda_2 \pm \left[(\mu_1 + 2\mu_3 N_0)^2\lambda_2^2 + 8\mu_2\mu_4\lambda_2\lambda_3|E_0|^2\right]^{\frac{1}{2}}\right\} \equiv \frac{\mu_1 + 2\mu_3 N_0}{2\mu_2} \pm \frac{\sqrt{\Delta_Q}}{2\mu_2\lambda_2}. \tag{17}$$

Therefore, $Q > 0$ for any K requires either

(i) that $\Delta_Q < 0$. This is only possible for the perturbation amplitudes E_0 and N_0 satisfying a specific relation. Thus, this case cannot be generally ensured for arbitrary perturbation amplitudes, or

(ii) that $\Delta_Q > 0$ and $K_{Q\pm}^2$ are both negative real values. It is ensured if $\mu_2(\mu_1 + 2\mu_3 N_0) < 0$ and $\mu_2\mu_4\lambda_2\lambda_3 < 0$; Otherwise,

if $\mu_2 > 0$, $\mu_1 + 2\mu_3 N_0 > 0$, $\mu_4\lambda_3 < 0$, and $\lambda_2 > 0$, then we have $K_{Q+}^2 > K_{Q-}^2 > 0$, and thus instability will arise when $K^2 \in (K_{Q-}^2, K_{Q+}^2)$;

if $\mu_2 < 0$, $\mu_1 + 2\mu_3 N_0 < 0$, $\mu_4\lambda_3 < 0$, and $\lambda_2 < 0$, then we have $K_{Q+}^2 > K_{Q-}^2 > 0$, and thus instability will arise when $K^2 \in (0, K_{Q-}^2) \cup (K_{Q+}^2, +\infty)$;

if $\mu_2 > 0$, $\mu_1 + 2\mu_3 N_0 > 0$, $\mu_4\lambda_3 > 0$, and $\lambda_2 < 0$, then we have $K_{Q-}^2 > K_{Q+}^2 > 0$, and thus instability will arise when $K^2 \in (K_{Q+}^2, K_{Q-}^2)$;

if $\mu_2 < 0$, $\mu_1 + 2\mu_3 N_0 < 0$, $\mu_4\lambda_3 > 0$, and $\lambda_2 > 0$, then we have $K_{Q-}^2 > K_{Q+}^2 > 0$, and thus instability will arise when $K^2 \in (0, K_{Q+}^2) \cup (K_{Q-}^2, +\infty)$;

if $\mu_2 > 0$, $\mu_1 + 2\mu_3 N_0 < 0$, $\mu_4\lambda_3 > 0$, and $\lambda_2 > 0$, then $K_{Q-}^2 < 0 < K_{Q+}^2$, and thus instability will arise when $K^2 \in (0, K_{Q+}^2)$;

if $\mu_2 < 0$, $\mu_1 + 2\mu_3 N_0 > 0$, $\mu_4\lambda_3 > 0$, and $\lambda_2 < 0$, then $K_{Q-}^2 < 0 < K_{Q+}^2$, and thus instability will arise when $K^2 \in (K_{Q+}^2, \infty)$;

if $\mu_2 > 0$, $\mu_1 + 2\mu_3 N_0 < 0$, $\mu_4\lambda_3 < 0$, and $\lambda_2 < 0$, then $K_{Q+}^2 < 0 < K_{Q-}^2$, and thus instability will arise when $K^2 \in (0, K_{Q-}^2)$; and

if $\mu_2 < 0$, $\mu_1 + 2\mu_3 N_0 > 0$, $\mu_4\lambda_3 < 0$, and $\lambda_2 > 0$, then $K_{Q+}^2 < 0 < K_{Q-}^2$, and thus instability will arise when $K^2 \in (K_{Q-}^2, \infty)$.

Finally, we have to check the last stability condition, $\Delta > 0$. Evidently, this condition is always satisfied when $Q < 0$. However, $\Delta > 0$ is ensured for any K if $\mu_4\lambda_2\lambda_3 > 0$, or in the cases that two roots for $\Delta = 0$, which are in the form of

$$K_{\Delta\pm}^2 = \frac{\mu_1 + 2\mu_3 N_0 \pm \sqrt{-8\lambda_2\lambda_3\mu_4|E_0|^2}}{\mu_2 - \lambda_2^2}, \tag{18}$$

have both negative values. Otherwise, instability will arise in the regions identified from (18) when one or both $K_{\Delta\pm}^2$ are positive depending on the parameters.

All the situations can be generally summarized as follows.

For $\Delta > 0$:

(i) If $P > 0$ and $Q > 0$, then $\Omega_{\pm}^2$ are both positive, therefore, we have four different real roots of (10);

(ii) If $Q < 0$, then $\Omega_-^2 < 0 < \Omega_+^2$, therefore, we have two different real roots and two nonreal complex (pure imaginary) conjugate roots of (10);

(iii) If $P < 0$ and $Q > 0$, then $\Omega_-^2 < \Omega_+^2 < 0$, therefore, we have four nonreal (two pure imaginary conjugate pairs of) roots of (10).

For $\Delta < 0$, we have four nonreal (two pure imaginary conjugate pairs of) roots of (10).

For $\Delta = 0$:

(i) If $P > 0$, we have two different real roots and one pair of pure imaginary conjugate roots of (10);

(ii) If $P < 0$, we have double pure imaginary roots of (10).

It is discovered that the SB system (1) – (2) might be modulationally unstable in several different unstable wave number regimes, either partially superimposed

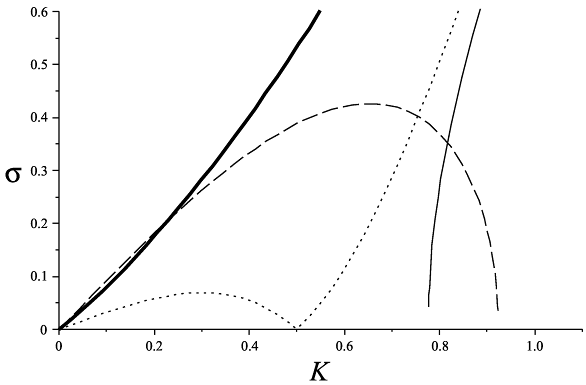

Fig. 1. Instability growth rate $\sigma = \sqrt{-\Omega_-^2}$ plotted versus the perturbation wave number K for the parameter values $\mu_4 = 1$, $|E_0| = 1/4$, and $\mu_2 = \lambda_2 = \mu_1 + 2\mu_3 N_0 = 1$, $\lambda_3 = -1$ (dashed line), $\mu_2 = -2$, $\lambda_2 = \lambda_3 = \mu_1 + 2\mu_3 N_0 = -1$ (dotted line), $\mu_2 = -1$, $\lambda_2 = -2$, $\lambda_3 = \mu_1 + 2\mu_3 N_0 = 1$ (dark line), and $\mu_2 = -2$, $\lambda_2 = 1$, $\lambda_3 = \mu_1 + 2\mu_3 N_0 = -1$ (solid line), respectively.

or distinct from each other. The growth rate σ of instability is given by the imaginary part $\mathrm{Im}(\Omega)$. In the case of $P < 0, Q < 0$, we have $\sigma = \sqrt{-\Omega_-^2}$ as a purely growing mode, and the corresponding wave number ranges can be determined by several situations, such as $(-K_{P-}, K_{P+}) \cap (0, K_{Q+})$, or $(-\infty, K_{P-}) \cap (K_{Q-}, K_{Q+})$, etc., depending on the parameter values. For given values of parameters and wave numbers, one can easily specify the growth rate. Figure 1 displays the growth rate in four particular sets of parameters.

3. Stationary Waves

Many approaches have been developed for finding exact solutions of nonlinear partial differential equations, such as the Darboux transformation, inverse scattering transformation, nonlinear variable separation approaches, and so on. Among them, the function expansion methods in various forms and generalizations are direct and powerful while relatively simple to command. Here, we look for solutions of the SB system through deforming them to the non-integrable ϕ^4 model, which possesses abundant known solutions. Moreover, some special Bäcklund transformations and nonlinear superpositions have been obtained for the $N+1$-dimensional ϕ^4 model [14], and thus more new solutions might be produced accordingly.

To look for stationary solitary waves of (1) and (2), we can write the complex E as $E = u\exp(\mathrm{i}kx - \mathrm{i}\omega t)$, where $u(x,t)$ is a real function, k and ω are real constants. Thus, (1)–(2) can be cast into the following system of three coupled equations:

$$\lambda_2 u_{xx} + (\lambda_4 - \lambda_3 N + \omega - k\lambda_1 - \lambda_2 k^2)u = 0, \quad (19)$$

$$u_t + (\lambda_1 + 2k\lambda_2)u_x = 0, \quad (20)$$

and

$$\begin{aligned} &N_{tt} + 2\mu_3 N_x^2 + (\mu_1 + 2\mu_3 N)N_{xx} - 2\mu_4 u_x^2 \\ &\quad + \mu_2 N_{xxxx} - 2\mu_4 u u_{xx} = 0. \end{aligned} \quad (21)$$

The general solution of (20) is in the form of

$$u = u(a(x - (2k\lambda_2 + \lambda_1)t)) \equiv u(\xi), \quad (22)$$

thus, (19) becomes

$$a^2\lambda_2 u_{\xi\xi} + (\lambda_4 - \lambda_3 N + \omega - k\lambda_1 - \lambda_2 k^2)u = 0, \quad (23)$$

and (21) can be rebuilt by $N(x,t) \equiv N(\xi)$ as

$$\begin{aligned} &((\lambda_1 + 2k\lambda_2)^2 + \mu_1 + 2\mu_3 N)N_{\xi\xi} + 2\mu_3 N_\xi^2 \\ &+ a^2\mu_2 N_{\xi\xi\xi\xi} - \mu_4(u^2)_{\xi\xi} = 0. \end{aligned} \quad (24)$$

It is easy to find that the following auto-Bäcklund transformation,

$$u = U_0 + U_1\phi + U_2\phi^2, \quad N = V_0 + V_1\phi + V_2\phi^2, \quad (25)$$

with undetermined constants U_i, V_i $(i = 0,1,2)$ can be used to deform the solutions of (23) and (24) to those of the ϕ^4 model [14–16]

$$\phi_\xi^2 = P\phi^4 + Q\phi^2 + R \quad (26)$$

with constants P, Q, and R.

Substituting (25) into (23) and (24), making use of (26), and then vanishing the coefficients of different powers of ϕ, a system of algebraic equations regarding the unknown constants is obtained, which has two classes of solutions.

Class I:

$$U_0 = U_2 = V_1 = 0, \; V_2 = -\frac{6\mu_2 a^2 P}{\mu_3}, \; \lambda_3 = -\frac{\mu_3\lambda_2}{3\mu_2}, \quad (27)$$

$$V_0 = -\frac{3\mu_2(a^2\lambda_2 Q + \lambda_4 + \omega - k\lambda_1 - \lambda_2 k^2)}{\mu_3\lambda_2}, \quad (28)$$

$$\begin{aligned} U_1 = \pm\{&6\mu_2 a^2 P[6(\lambda_4 + \omega)\mu_2 \\ &-2k(2\lambda_2^2 + 3\mu_2)(\lambda_2 k + \lambda_1) \\ &+ \lambda_2(2Q\mu_2 a^2 - \mu_1 - \lambda_1^2)]/\mu_3\mu_4\lambda_2\}^{1/2}. \end{aligned} \quad (29)$$

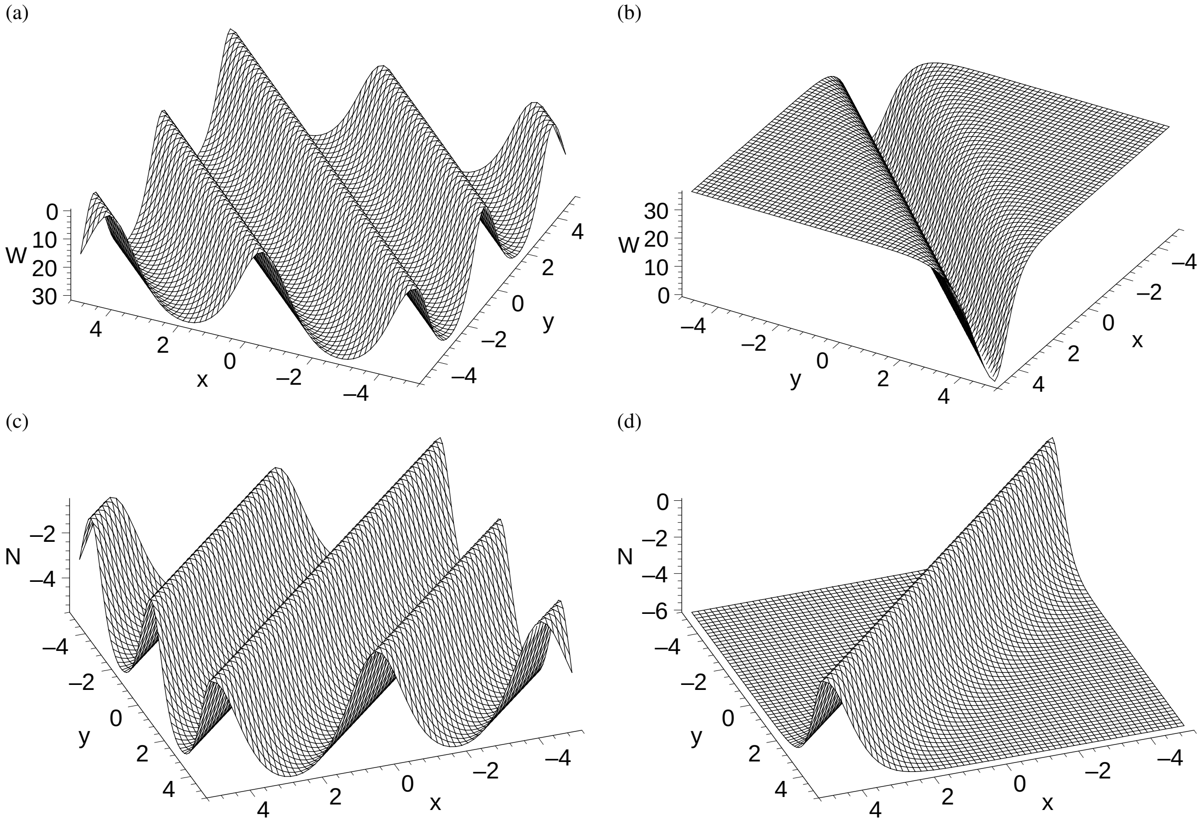

Fig. 2. Profiles of periodic wave solutions of (a) $W \equiv |E|^2$ and (c) N given by (34) and (35), respectively, with the parameters (39). The waves in the long wave limit under $m \to 1$ are shown in (b) and (d), correspondingly.

Class II:

$$U_1 = V_1 = 0, \quad V_2 = \frac{6\lambda_2 a^2 P}{\lambda_3}, \\ U_2 = \pm\frac{6a^2 P}{\lambda 3}\sqrt{\frac{\lambda_2(\mu_2\lambda_3 + \lambda_2\mu_3)}{\mu_4}}, \tag{30}$$

$$U_0 = \pm\frac{2a^2(Q + \delta\sqrt{Q^2 - 3PR})}{\lambda_3} \\ \cdot\sqrt{\frac{\lambda_2(\mu_2\lambda_3 + \mu_3\lambda_2)}{\mu_4}}, \tag{31}$$

$$V_0 = -\frac{1}{\lambda_3}\Big(2\lambda_2 a^2\delta\sqrt{Q^2 - 3PR} + \lambda_4 \\ - k\lambda_1 - \lambda_2 k^2 + \omega + 2a^2\lambda_2 Q\Big), \tag{32}$$

$$\mu_1 = \frac{1}{\lambda_3}\Big[2k(\mu_3 - 2\lambda_2\lambda_3)(k\lambda_2 + \lambda_1) - \lambda_3\lambda_1^2 \\ - 2\mu_3\lambda_4 - 2\mu_3\omega\Big], \tag{33}$$

with $\delta^2 = 1$.

Therefore, many solitary waves can be explored via the abundant waves of the model (26). In the following, we list three examples.

Example 1. It is known that (26) with $P = m^2$, $Q = -1 - m^2$, and $R = 1$ has the solution $\phi = \mathrm{sn}(\xi, m)$. In this case, the solutions of the SB system can be obtained as

$$E = \pm\Big\{6\mu_2 a^2 m^2[6(\lambda_4 + \omega)\mu_2 \\ - 2k(2\lambda_2^2 + 3\mu_2)(\lambda_2 k + \lambda_1) \\ - \lambda_2(2(1 + m^2)\mu_2 a^2 + \mu_1 + \lambda_1^2)]/\mu_3\mu_4\lambda_2\Big\}^{\frac{1}{2}} \\ \cdot\,\mathrm{sn}(\xi, m)\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}, \tag{34}$$

$$N = \frac{3\mu_2(a^2\lambda_2(1 + m^2) - \lambda_4 - \omega + k\lambda_1 + \lambda_2 k^2)}{\mu_3\lambda_2} \\ - \frac{6\mu_2 a^2 m^2}{\mu_3}\mathrm{sn}^2(\xi, m), \tag{35}$$

with $\xi = a(x - (2k\lambda_2 + \lambda_1)t)$ and the condition $3\mu_2\lambda_3 + \mu_3\lambda_2 = 0$, which are also valid to the follow-

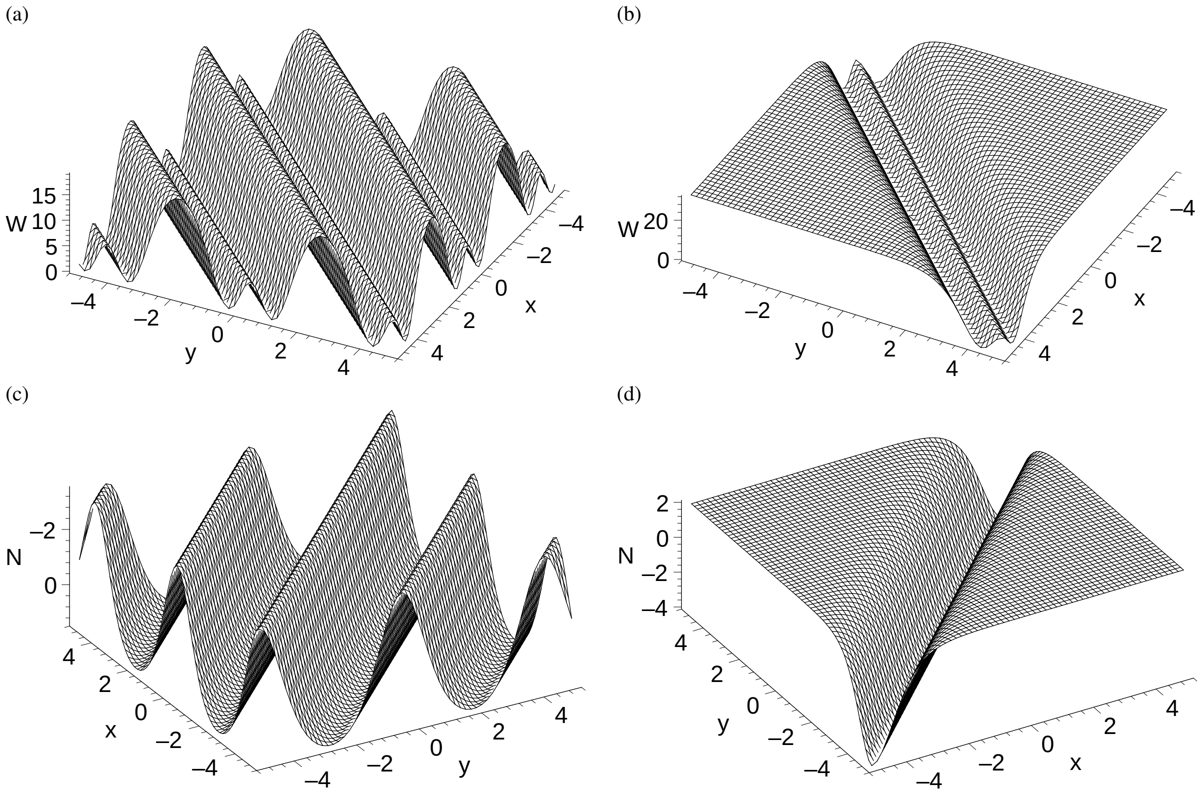

Fig. 3. Profiles of periodic wave solutions of (a) $W \equiv |E|^2$ and (b) N given by (36) and (37) with the parameters (40), respectively. The waves in the long wave limit under $m \to 1$ are shown in (b) and (d), correspondingly.

ing cases and thus will not be declared again below, and

$$E = \pm \frac{2a^2}{\lambda_3}\sqrt{\frac{\lambda_2(\mu_2\lambda_3+\lambda_2\mu_3)}{\mu_4}} \cdot \left[c\delta - m^2 - 1 + 3m^2\mathrm{sn}^2(\xi,m)\right] \mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}, \tag{36}$$

$$N = -\frac{1}{\lambda_3}\Big[2a^2\lambda_2(c\delta - 4m^2 - 1) + \lambda_4 - k\lambda_1 - \lambda_2 k^2 + \omega\Big]\mathrm{sn}^2(\xi,m), \tag{37}$$

with $\delta^2 = 1$, $c = \sqrt{1-m^2+m^4}$ and the condition

$$2k(2\lambda_2\lambda_3 - \mu_3)(k\lambda_2 + \lambda_1) + \lambda_3(\mu_1 + \lambda_1^2) + 2\mu_3(\lambda_4 + \omega) = 0, \tag{38}$$

which are also valid to the following cases and thus will not be pointed out again below.

Representative wave structures of $W \equiv |E|^2$ and N determined by (34) and (35), respectively, are displayed in Figure 2 with the parameters

$$a = k = \lambda_2 = \lambda_4 = \omega = \mu_1 = \mu_2 = \mu_3 = \mu_4 = 1, \quad \lambda_1 = -1, \quad m = 0.9. \tag{39}$$

It is seen from Figures 2a and 2c that the high and low frequency modes have similar periodic wave profiles, while in the limit of modulus m approaching unity, they show opposite behaviours, namely, becoming dark and bright solitary waves, respectively, as shown in Figures 2b and 2d. Figure 3 shows the wave profiles of $W \equiv |E|^2$ and N determined by (36) and (37), respectively, with the parameters

$$a = k = \delta_1 = \lambda_2 = \lambda_3 = \lambda_4 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = 1, \quad \lambda_1 = -1, \quad m = 0.9. \tag{40}$$

It is observed from this figure, different from the previous case, that the two modes now possess different periodic waves as shown in Figures 3a and 3c. In addition, when m goes to unity, N turns into a simple dark solitary wave, while W is a 'W' shaped wave, exhibited in Figures 3b and 3d.

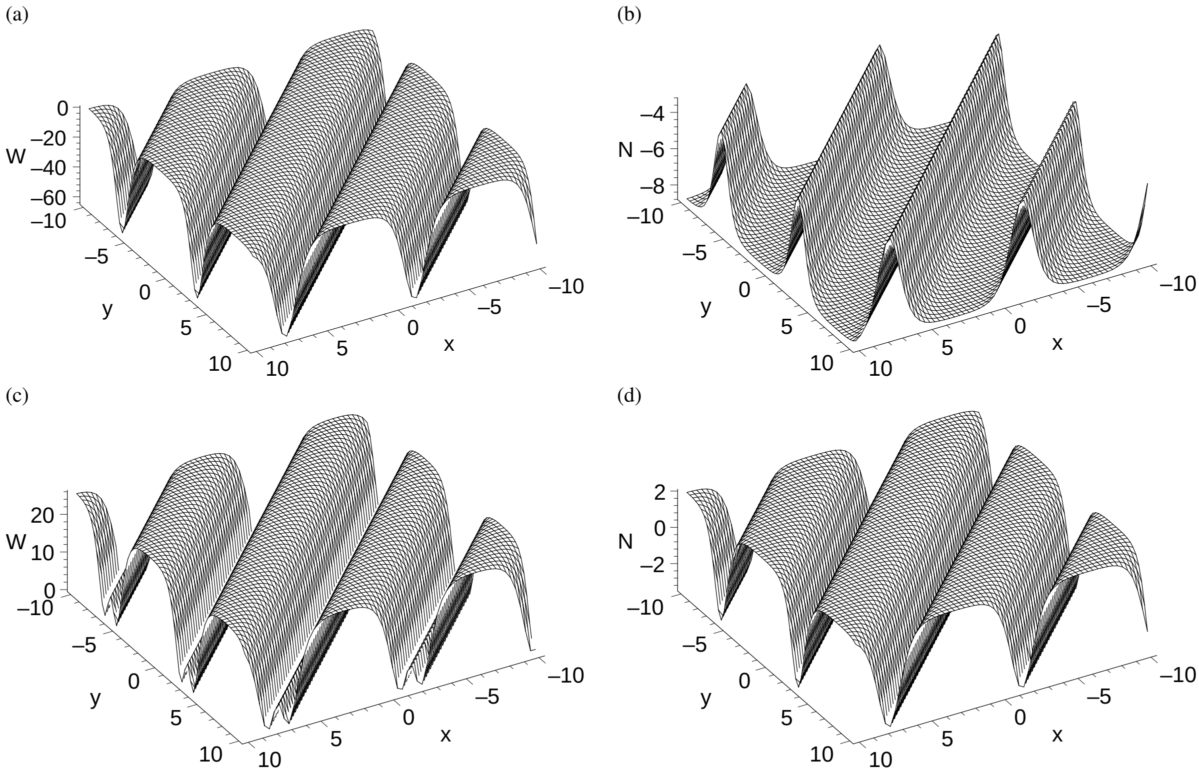

Fig. 4. Profiles of periodic wave solutions of (a) $W \equiv |E|^2$ and (b) N given by (45) and (46) with the parameters (39), respectively; (c) $W \equiv |E|^2$ and (d) N given by (47) and (48) with the parameters (40), respectively.

Example 2. If we suppose $P = R = m^2/4$ and $Q = -1 + m^2/2$, then the ϕ^4 equation (26) possesses the solution $\phi = m\ \mathrm{sn}(\xi,m)/(1+\mathrm{dn}(\xi,m))$. In this case, the exact solutions of (1)–(2) are in the form of

$$\begin{aligned}E = \pm\Big\{&3\mu_2 a^2 m^2[6\mu_2(\lambda_4+\omega) \\ &+\mu_2\lambda_2 a^2(m^2-2)-\lambda_2(\mu_1+\lambda_1^2) \\ &-2k(2\lambda_2^2+3\mu_2)(\lambda_2 k+\lambda_1)]/2\mu_3\mu_4\lambda_2\Big\}^{\frac{1}{2}} \\ &\cdot\frac{m\ \mathrm{sn}(\xi,m)}{1+\mathrm{dn}(\xi,m)}\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t},\end{aligned} \quad (41)$$

$$\begin{aligned}N = &\frac{3\mu_2 a^2\lambda_2(2-m^2)-6\mu_2(\lambda_4+\omega-k\lambda_1-\lambda_2 k^2)}{2\mu_3\lambda_2} \\ &-\frac{3\mu_2 a^2 m^3\mathrm{sn}^2(\xi,m)}{2\mu_3(1+\mathrm{dn}(\xi,m))^2},\end{aligned} \quad (42)$$

and

$$\begin{aligned}E = \pm&\frac{a^2}{\lambda_3}\sqrt{\frac{\lambda_2(\mu_2\lambda_3+\lambda_2\mu_3)}{\mu_4}} \\ &\cdot\left[\frac{\delta}{2}\sqrt{2(m^2+4)(2-m^2)}+m^2\right. \\ &\left.-2+\frac{3m^4\mathrm{sn}^2(\xi,m)}{2(1+\mathrm{dn}(\xi,m))^2}\right]\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t},\end{aligned} \quad (43)$$

$$\begin{aligned}N = -&\frac{1}{2\lambda_3}\left[\lambda_2 a^2\left(\delta\sqrt{2(m^2+4)(2-m^2)}-m^2-4\right)\right. \\ &\left.+2(\lambda_4-k\lambda_1-\lambda_2 k^2+\omega)\right]\frac{m^2\mathrm{sn}^2(\xi,m)}{(1+\mathrm{dn}(\xi,m))^2}.\end{aligned} \quad (44)$$

Example 3. Equation (26) has the solution $\phi = \mathrm{dn}(\xi,m)/(1+m\ \mathrm{sn}(\xi,m))$ when $P = R = (m^2-1)/4, Q = (m^2+1)/2$. Therefore, the corresponding solutions of the SB system are determined to be

$$\begin{aligned}E = \pm\Big\{&3\mu_2 a^2(m^2-1)[6(\lambda_4+\omega)\mu_2 \\ &-\lambda_2(\mu_1+\lambda_1^2-\mu_2 a^2(m^2+1)) \\ &-2k(2\lambda_2^2+3\mu_2)(\lambda_2 k+\lambda_1)]/2\mu_3\mu_4\lambda_2\Big\}^{\frac{1}{2}} \\ &\cdot\frac{\mathrm{dn}(\xi,m)}{1+m\ \mathrm{sn}(\xi,m)}\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t},\end{aligned} \quad (45)$$

$$N = -\frac{3\mu_2 a^2 \lambda_2 (m^2+1) + 6\mu_2(\lambda_4 + \omega - k\lambda_1 - \lambda_2 k^2)}{2\mu_3\lambda_2} - \frac{3\mu_2 a^2 (m^2-1)\mathrm{dn}^2(\xi,m)}{2\mu_3(1+m\,\mathrm{sn}(\xi,m))^2}, \tag{46}$$

and

$$E = \pm\frac{a^2}{2\lambda_3}\sqrt{\frac{\lambda_2(\mu_2\lambda_3+\lambda_2\mu_3)}{\mu_4}} \cdot \left[\delta\sqrt{1+14m^2+m^4} + 2(m^2+1) + \frac{3(m^2-1)\mathrm{dn}^2(\xi,m)}{(1+m\,\mathrm{sn}(\xi,m))^2}\right]\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}, \tag{47}$$

$$N = -\frac{1}{2\lambda_3}\left[\lambda_2 a^2(\delta\sqrt{1+14m^2+m^4} - m^2 + 5) + 2(\lambda_4 - k\lambda_1 - \lambda_2 k^2 + \omega)\right]\frac{\mathrm{dn}^2(\xi,m)}{(1+m\,\mathrm{sn}(\xi,m))^2}. \tag{48}$$

Representative profiles of the periodic wave solutions given by (45) and (46) with parameters (39), and by (47) and (48) with parameters (40), respectively, are displayed in Figure 4. However, in this case, the waves will disappear (i. e., having constant W and N) in the long wave limit $m \to 1$.

It is remarkable that in some cases, some solutions expressed by the Jacobian elliptic functions have singularities and thus they might blow up at some points. Nonetheless, in the long wave limit the modulus approaching unity, these solutions become regular. For instance, when $P = R = 1/4$ and $Q = (1-2m^2)/2$, (26) has a singular solution in the form of $\phi = (1-\mathrm{cn}(\xi,m))/\mathrm{sn}(\xi,m)$, which can lead to another type of solutions for the SB system. It is obvious that in the limit $m \to 1$, this solution turns into $\phi = (1-\mathrm{sech}(\xi))/\tanh(\xi)$, and thus one can obtain a regular wave solution of the SB equation.

4. Summary

The modulational instability of the generalized Schrödinger-Boussinesq system is investigated in detail. The nonlinear dispersion relation associated with the perturbation wave number is discovered. Due to the intrusion of many parameters in the model, many possibilities exist for instabilities might arise in different regions of the perturbed wave number, as observed from Figure 1 in which four cases are displayed.

A Bäcklund transformation in a polynomial form between the SB system and the ϕ^4 model can be straightforwardly derived via the Painlevé truncation method. In such a way, abundant solutions of the ϕ^4 model can act as the building blocks of the solutions of the SB system. Two classes of solutions in general forms are first obtained, followed by three types of explicit periodic wave solutions. It is shown that the high-frequency and the low-frequency modes can have rich nonlinear wave excitations either of the same or different depending on the parameters. Some representative profiles are graphically displayed.

Since the SB system can model an amplitude modulated high-frequency wave coupled to a suitable low-frequency eigenmode in nonlinear dispersive media like plasma, our solutions might be useful to describe those associated nonlinear phenomena.

[1] N. N. Rao, Phys. Scripta **T63**, 219 (1996).
[2] N.N. Rao and P.K. Shukla, Phys. Plasmas **4**, 636 (1997).
[3] N. L. Shatashvili and N. N. Rao, Phys. Plasmas **6**, 66 (1999).
[4] P. K. Shukla, N. N. Rao, M. Y. Yu, and N. L. Tsintsadze, Phys. Rep. **138**, 1 (1986).
[5] X. B. Hu, B. L. Guo, and H. W. Tam, J. Phys. Soc. Jpn. **72**, 189 (2003).
[6] Y. Hase and J. Satsuma, J. Phys. Soc. Jpn. **57**, 679 (1988).
[7] Y.C. Hon and E.G. Fan, Nonlinear Anal.: Theory, Methods Appl. **71**, 3501 (2009).
[8] A. R. Chowdhury, B. Dasgupta, and N. N. Rao, Chaos, Solitons, and Fractals **9**, 1747 (1998).
[9] S. Hakkaev, Commun. Pure Appl. Anal. **6**, 1043 (2007).
[10] M. Marklund, P. K. Shukla, B. Bingham, and J. T. Mendonca, Phys. Rev. E **74**, 067603 (2006).
[11] X. Y. Tang and P. K. Shukla, J. Phys. A: Math. Theor. **40**, 3729 (2007).
[12] X. Y. Tang and W. Ding, Phys. Scr. **77**, 015004 (2008).
[13] E. Yomba and G.R. Sell, J. Math. Phys. **50**, 053518 (2009).
[14] M. Jia, F. Huang, and S. Y. Lou, Phys. Scr. **73**, 368 (2006).
[15] A. Darwish and E. G. Fan, Chaos, Solitons, and Fractals **20**, 609 (2004).
[16] S. Y. Lou, H. C. Hu, and X. Y. Tang, Phys. Rev. E **71**, 036604 (2005).